2.3a local stability of first order equations

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Lemma (Translation): Let $\bar{x}$ be an equilibrium of $x_{t+1} = f(x_t)$.

Define a variable $u_t = x_t - \bar{x}$. Then $\bar{u} = 0$ is an equilibrium of $u_{t+1} = g(u_t)$, where $g(u) = f(u + \bar{x}) - f(\bar{x})$. Furthermore,

0 is locally stable (or unstable, or locally asymptotically stable) fixed pt of g

iff $\bar{x}$ " " " " " " " " " " " " " f.

Consider: Suppose $\exists f''$ continuous on an open interval $I \ni \bar{x}$. Then by Taylor's thm,

$$f(x) = f(\bar{x}) + f'(\bar{x})(x - \bar{x}) + \frac{f''(\xi)}{2!}(x - \bar{x})^2$$

for some $\xi \in I$. If $(x_t - \bar{x})$ is small, can approximate

$$\left.\begin{aligned} f(x_t) - \bar{x} &\approx f'(\bar{x})(x - \bar{x}) \\ \text{or} \quad u_{t+1} &\approx f'(\bar{x})\, u_t \end{aligned}\right\} \text{linear approximation at } \bar{x}$$

Thm 2.1 Let f have a continuous first derivative f' on an open interval $I \ni \bar{x}$, and $\bar{x}$ is a fixed pt of f.

Then $\bar{x}$ is a locally asymptotically stable equilibrium of $x_{t+1} = f(x_t)$ if

$$|f'(\bar{x})| < 1$$

and unstable if $|f'(\bar{x})| > 1$.

proof. Case 1: $|f'(\bar{x})| < 1$. Because f' is continuous on I, can choose

$$[\bar{x} - \varepsilon, \bar{x} + \varepsilon] \subset I \text{ s.t. } |f'(x)| < c < 1 \text{ for } x \in [\bar{x} - \varepsilon, \bar{x} + \varepsilon].$$

By the Mean Value Thm (MVT) $\forall x_0 \in [\bar{x} - \varepsilon, \bar{x} + \varepsilon]$,

$$|\bar{x} - \underbrace{f(x_0)}_{x_1}| = |f(\bar{x}) - f(x_0)| = |f'(\xi_1)|\,|\bar{x} - x_0| \le c\,|\bar{x} - x_0|.$$

MVT, ξ_1 between $\bar{x}$ and x_0 so $\xi_1 \in [\bar{x} - \varepsilon, \bar{x} + \varepsilon]$

Suppose $|\bar{x} - f(x_{t-1})| \le c|\bar{x} - x_{t-1}|$ and $x_{t-1} \in [\bar{x}-\varepsilon, \bar{x}+\varepsilon]$.

First, $x_t = f(x_{t-1}) \in [\bar{x}-\varepsilon, \bar{x}+\varepsilon]$ because $c<1$.

Then $|\bar{x} - f(x_t)| = |f(\bar{x}) - f(x_t)| = |f'(\underbrace{\xi_{t+1}}_{\text{between } \bar{x} \text{ and } x_t})| \, |\bar{x} - x_t| \le c|\bar{x} - x_t|$.

Then by induction, $|\bar{x} - f(x_t)| \le c^t |\bar{x} - x_0|$

$\Rightarrow \lim_{t\to\infty} x_t = \bar{x}$, so $\bar{x}$ is locally asymptotically stable.

<u>Case 2:</u> $|f'(\bar{x})| > 1$.

Then $\exists\, \varepsilon > 0$ s.t. for $x \in [\bar{x}-\varepsilon, \bar{x}+\varepsilon] \subset I$, $|f'(x)| > c > 1$.

By the MVT,

$$|\bar{x} - f(x_0)| = |f'(\underbrace{\xi_1}_{\text{between } \bar{x} \text{ and } x_0})| \, |\bar{x} - x_0| \ge c|\bar{x} - x_0|.$$

But this time, if we try to use induction, eventually $c^t|\bar{x} - x_0| > \varepsilon$.

Hence, $\exists\, t$ s.t. $|\bar{x} - f^t(x_0)| > \varepsilon$, so $\bar{x}$ is unstable. $\blacksquare$

<u>Def. 2.4</u> An equilibrium $\bar{x}$ of $x_{t+1} = f(x_t)$ is **hyperbolic** if

$|f'(\bar{x})| \ne 1$ and **nonhyperbolic** otherwise.

<u>Aside:</u> We can also generalize notions of stability to periodic solutions of period m by considering the function $f^m(x)$ instead of $f(x)$ in Thm 2.1.

<u>Thm 2.2</u> Suppose f' is continuous on an open interval I and the m-cycle $\{\bar{x}_1, f(\bar{x}_1), \ldots, f^{m-1}(\bar{x}_1)\} \subset I$.

$\{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_m\}$

Then the m-cycle is locally asymptotically stable if ~~for~~ some k

$$\left|\frac{d}{dx} f^m(\bar{x}_k)\right| < 1$$

and unstable if for some k

$$\left| \frac{d}{dx} f^m(\bar{x}_k) \right| > 1.$$

Corollary 2.1 Suppose $\{\bar{x}_1, \dots, \bar{x}_m\}$ is an m-cycle of the difference eq $x_{t+1} = f(x_t)$. Then the m-cycle is asymptotically stable if

$$|f'(\bar{x}_1) \cdots f'(\bar{x}_m)| < 1.$$

Next time: What about the nonhyperbolic case? We can't ignore higher-order terms